Detecting chaos from time series

2000 J. Phys. A: Math. Gen. 33 1007

(http://iopscience.iop.org/0305-4470/33/5/313)

J. Phys. A: Math. Gen. **33** (2000) 1007–1016. Printed in the UK PII: S0305-4470(00)03642-8

Detecting chaos from time series

Gong Xiaofeng and C H Lai
Department of Physics, National University of Singapore, Singapore 119 260, Singapore
E-mail: scip7254@nus.edu.sg and phylaich@nus.edu.sg

Received 19 April 1999, in final form 11 November 1999

Abstract. In this paper, an entirely data-based method to detect chaos from the time series is developed by introducing ϵ^p-neighbour points (the p-steps ϵ-neighbour points). We demonstrate that for deterministic chaotic systems, there exists a linear relationship between the logarithm of the average number of ϵ^p-neighbour points, $\ln n_{p,\epsilon}$, and the time step, p. The coefficient can be related to the KS entropy of the system. The effects of the embedding dimension and noise are also discussed.

Over the last decades chaotic behaviour of dynamical processes has been reported in many scientific fields [1, 2]. Chaotic dynamics appears to provide a relatively simple and possibly more satisfactory explanation of complex phenomena. However, it must be seriously doubted whether there are chaotic attractors underlying all these systems. Consequently, the classification of dynamical systems that one observes is a critical part of the analysis of measured signals. In real experiments, usually only short time series, often distorted by measurement errors, are available. It makes the numerical extraction of physically relevant quantities such as the spectrum of Lyapunov exponents, the Kolmogorov–Sinai (KS) entropy or metric entropy, and the fractal dimension a difficult task. Often the reliability of the results is also not guaranteed. In general, the two major features which are used widely as classifiers of chaotic systems are the fractal dimensions and Lyapunov exponents [3]. Fractal dimensions are characteristic of the geometry of the attractor, and relate to the way points on the attractor are distributed in the d_E-dimensional space (d_E refers to the embedding dimension). The Lyapunov exponents serve to indicate how orbits on the attractor move apart (or together) under the evolution of the dynamics. While a number of algorithms for detecting nonlinearity of time series have been proposed (see [8, 14] and references therein), problems still exist. For example, if an algorithm for estimating the Lyapunov spectra is blindly applied to a time series that has a stochastic origin, it has been shown that spurious positive Lyapunov exponents would be obtained [4]. There is thus a strong demand for new practical and reliable methods as the complementary tools, which allows the determination of the nature of a given time series, especially for a data set contaminated by noise. Although any single method may not be totally conclusive, it will give us greater confidence about the nature of the data if the conclusions obtained from different methods are similar. From the theoretical point of view, the KS entropy is the most appropriate quantity to differentiate different classes of processes: the KS entropy is zero for periodic processes, finite and positive for chaotic systems and infinitive for stochastic processes. Of course, there will still be potential problems when real-world time series data are considered; for example, if noisy chaos is observed, the calculated KS entropy should be

0305-4470/00/051007+10$30.00

infinite if the scale is small enough [18]. In this paper, we suggest a new method to identify the chaotic data by investigating the average number of the ϵ^p-neighbours. The method is entirely based on the data set, and can easily be applied to any time series without resorting to Poincaré maps. We will argue that this measure is closely related to the KS entropy of the underlying dynamics. Furthermore, this method is robust with respect to low-level noise, and can even be used to provide a rough estimate of the ratio of signal-to-noise. The method is established by considering the local properties of the point elements in the chaotic attractors.

Suppose we have an observed time series $\{s(k), k = 1, 2, \ldots, N\}$. Using the standard time-delay method [5–7], we can reconstruct the state space of the underlying system with the d-dimensional delay-coordinate vectors $y_k = (s(k), s(k+\tau), \ldots, s(k+(d-1)\tau))$, where τ is the time delay, and d denotes the sufficiently high embedding dimension (e.g. $d > 2D_0$, where D_0 is the box counting dimension of the system). Now consider a typical point in the attractor of the system y_k, and define the ϵ-neighbourhood of y_k as

$$\Omega(y_k, \epsilon) = \{y_j | \parallel y_j - y_k \parallel < \epsilon, j \neq k\} \tag{1}$$

where the distance takes, for computational efficiency, the following form:

$$\parallel y_k - y_{k'} \parallel = \max\{| s(k+i\tau) - s(k'+i\tau) |, i = 0, 1, \ldots, d-1\}. \tag{2}$$

Let $n(y_k, \epsilon)$ be the number of the ϵ-neighbours of y_k (we also call such points the ϵ^0-neighbours, for a reason that will become clear later). After one iteration, y_k is mapped to $y_{k+1} = f(y_k)$, and the y_k ϵ-neighbours $y_j \in \Omega(y_k, \epsilon)$ are mapped to $f(y_j)$ for every j. Suppose the ϵ-neighbourhood of y_{k+1} is $\Omega(y_{k+1}, \epsilon)$. We now consider the set

$$\Omega^1(y_k, \epsilon) = \{y_j | y_j \in \Omega(y_k, \epsilon) \cap \Omega(y_{k+1}, \epsilon)\}. \tag{3}$$

Obviously, any point $y_j \in \Omega^1(y_k, \epsilon)$ satisfies the following conditions simultaneously:

$$\begin{aligned} &\parallel y_j - y_k \parallel < \epsilon \\ &\parallel f(y_j) - f(y_k) \parallel < \epsilon. \end{aligned} \tag{4}$$

Since these points will keep close to y_k one step into future, we call such points the ϵ^1-neighbours of y_k, and let $n(y_{k+1}, \epsilon)$ denotes the number of these points. Continuing in this manner, we can define the ϵ^p-neighbourhood of y_k as

$$\begin{aligned} \Omega^p(y_k, \epsilon) &= \{y_j | y_j \in \Omega(y_k, \epsilon) \cap \Omega(y_{k+1}, \epsilon) \cdots \cap \Omega(y_{k+p}, \epsilon), j \neq k\} \\ &= \{y_j | \parallel y_{j+i} - y_{k+i} \parallel < \epsilon, i = 0, 1, 2, \ldots, p, j \neq k\} \end{aligned} \tag{5}$$

and the number of the ϵ^p-neighbours is $n(y_{k+p}, \epsilon)$, where $p = 1, 2, \ldots$. The number of the ϵ^p-neighbours $n(y_{k+p}, \epsilon)$ reflects the local property of the system. For a chaotic system, in fact, this quantity is related to the KS entropy of the system. To show this, let us consider for simplicity the two-dimensional hyperbolic chaotic map. In this case, the phase space can be partitioned into stable and unstable manifolds. Suppose at time $t + p$, the (square) ϵ-neighbourhood of a typical point x_{t+p} in the attractor is $S(x_{t+p}, \epsilon \times \epsilon)$ (see figure 1), where we have chosen the sides of the square to be along the stable and unstable manifold, respectively. Now let x_t be the p-step pre-image of x_{t+p}. If ϵ is small enough, the p-step pre-image of the neighbourhood $S(x_{t+p}, \epsilon \times \epsilon)$ can be approximately written as $S(x_t, \epsilon e^{-p\lambda_1} \times \epsilon e^{-p\lambda_2})$, where $S(x_t, \epsilon e^{-p\lambda_1} \times \epsilon e^{-p\lambda_2})$ represents the rectangle neighbourhood of x_t, and the lengths of the two sides are $\epsilon e^{-p\lambda_1}$ and $\epsilon e^{-p\lambda_2}$ (corresponding to the vertical thick framed slice on the left of figure 1). Here $\lambda_1 > 0$ and $\lambda_2 < 0$ are the Lyapunov exponents of the system. According to the definition above (in equation (5)), the ϵ^p-neighbourhood of x_t should be distributed within the area $S(x_t, \epsilon e^{-p\lambda_1} \times \epsilon)$ (corresponding to the shadowed part in figure 1). On the other hand, if we take local fractality into account, we get

$$\begin{aligned} n(x_t, \epsilon) &\sim \epsilon^{D_1} \cdot \epsilon^{D_2} \\ &\sim \epsilon^{D_1 + D_2} \end{aligned} \tag{6}$$

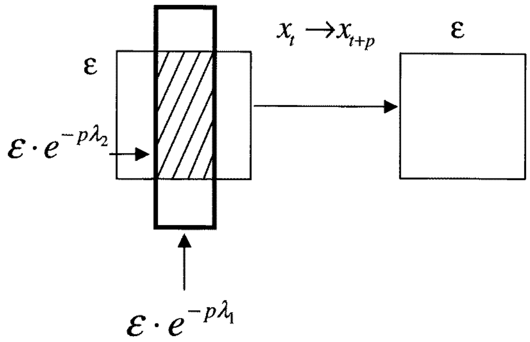

Figure 1. A schematic diagram illustrating the relationship between the number of the ϵ^p-neighbours, $n(y_{t+p}, \epsilon)$, and the dynamical quantities of the system.

and

$$n(x_{t+p}, \epsilon) \sim (\epsilon e^{-p\lambda_1})^{D_1} \cdot \epsilon^{D_2} \\ \sim n(x_t, \epsilon) e^{-p\lambda_1 D_1} \tag{7}$$

where D_i ($i = 1, 2$) are called partial dimensions and are the information dimensions of the sets formed by the intersection of the attractor with the unstable or stable directions related to λ_i [15, 19]. Using the generalized Pesin identity for KS entropy [15, 19]

$$h_{KS} = \sum_{i;\lambda_i>0} D_i \lambda_i. \tag{8}$$

Equation (7) can then be rewritten as

$$n(x_{t+p}, \epsilon) \sim n(x_t, \epsilon) e^{-p h_{KS}}. \tag{9}$$

It should be stated that the above derivation cannot be considered as a rigorous proof; a formal mathematical theorem about this result can be found in [20]. In fact, it is equation (8) (the Ledrappier–Young formula) which has to be deduced from equations (7) and (9). However, it is useful to understand the relationship between the ϵ^p-neighbour points and the invariant quantities of the underlying chaotic systems. To determine an invariant under the evolution of the system, we need to take the average of $n(y_{k+p}, \epsilon)$ and normalize it by the total numbers of data points. Averaging over the attractor, we can define the average number of ϵ^p-neighbour points of the attractor as

$$n_{p,\epsilon} = \frac{1}{N-p} \sum_{k=1}^{N-p} n(y_{k+p}, \epsilon). \tag{10}$$

We can also express the average number of the ϵ^p neighbours $n_{p,\epsilon}$ in another way, which shows the relationship between $n_{p,\epsilon}$ and the so-called correlated sum [9]. Suppose we rewrite $n_{p,\epsilon}$ as follows: for any y_k, $k \leqslant N - p$

$$n(y_k, \epsilon) = \sum_{\substack{j=1 \\ j \neq k}}^{N} \theta(\epsilon - \| y_k - y_j \|) \tag{11}$$

where $\theta(x)$ is the Heaviside function. By $n(y_{k+p}, \epsilon)$, we mean the number of points that always stay within the ϵ-neighbour until the pth step. Thus, for $k \leqslant N - p$, we have

$$n(y_{k+p}, \epsilon) = \sum_{\substack{j=1 \\ j \neq k}}^{N-p} \prod_{i=0}^{p} \theta(\epsilon - \| y_{k+p} - y_{j+p} \|). \tag{12}$$

After averaging, we obtain

$$n_{p,\epsilon} = \frac{1}{N-p} \sum_{\substack{k,j=1 \\ j \neq k}}^{N} \prod_{i=0}^{p} \theta(\epsilon - \| y_{k+p} - y_{j+p} \|). \tag{13}$$

Note that this is different from the well known correlation sum [8, 9]:

$$C(q, \epsilon) = \frac{1}{N} \sum_{k=1}^{N} \left\{ \frac{1}{N} \sum_{j=1}^{N} \theta(\epsilon - \| y_k - y_j \|) \right\}^{q-1} \tag{14}$$

which relates more to the distribution of the points on the whole attractor. Our approach places greater emphasis on the dynamical process of the system. As mentioned above, $n_{p,\epsilon}$ describes the average of certain local property of the dynamical system. Since the most prominent characteristic of a chaotic system is the exponential divergence of nearby orbits, we expect to be able to identify the chaotic system by the way of the evolution of $\ln n_{\epsilon,p}$ with p and ϵ, based on equation (9). For a totally random system, since there is no deterministic relationship between the consecutive points in an orbit, we can expect there is no such relationship between these quantities. (We shall discuss in greater detail the case with independent uniformly distributed noise later.) For practical calculation, however, the two parameters N (the length of the time series under consideration) and ϵ should be considered carefully. First, we shall assume N is large enough, which means the time series covers all of the important structures of the underlying system, e.g., for a Lorenz system, we need the time series to at least visit both the two lobes. This is to ensure that the average number of the ϵ^p-neighbours reflects the property of the whole system and not just a part of the system. In practical applications, it is difficult to know whether the number of the data points is indeed 'large enough' before we know something about the system. The reasonable approach is then to investigate the evolution of $n_{p,\epsilon}$ with N. If for some N_0 there exists a stable value of $n_{\epsilon,p}$ which is not sensitive to further increase of the number of data points, then a value slightly larger than N_0 can be considered as the acceptable value. In our numerical simulations reported here, we utilize all of the available data points since our purpose is to examine if the method actually works as expected. Strictly speaking, equation (9) is satisfied only in the sense of infinitesimally small ϵ. In practical cases, however, this is clearly impossible due to the finite length of the time series. We shall assume that if ϵ is small enough, equation (9) will still give us a good approximation. This means that, for a given N, we can find a range (ϵ_b, ϵ_u) of values of ϵ which is large enough (e.g. $\epsilon_u/\epsilon_b \sim 10$), and the evolution of the slope of $\ln n_{p,\epsilon} \sim p$ is almost invariant in this range. We then take this value as the estimate of h_{KS}. In practice, we choose ϵ large enough to contain a sufficient number of points within the ϵ-neighbourhood, and still be small compared with the magnitude of the signal (between 1% and 10% of the attractor size). Our numerical results show that, for many chaotic systems, a wide range of values of ϵ can work well to capture the characteristic of the chaotic time series. It should be pointed out that the choices of N and ϵ are not, strictly speaking, independent, and we see no clear objective criterion to determine just what the best ϵ value is. We only expect that it is dependent on the dimension of the underlying chaotic system, the length of the time series (perhaps even the sample rate), and the size of the attractor. But then this is a difficulty that every algorithm to detect chaos from time series data suffers from. We have applied the above method to several systems. As the simplest example, we consider the data set generated by the logistic map

$$y_{n+1} = \lambda y_n (1 - y_n) \tag{15}$$

where $\lambda = 3.999$. The result is displayed in figure 2(*a*). It is clear that for a wide range of p and ϵ, $\ln n_{p,\epsilon}$ shows a linear relationship with p, and the slope is about 0.65, reasonably close to the KS entropy of the system. We believe the difference is caused mainly by the finite extent N of the data (only 1000 points are used), and we cannot get smaller ϵ values because the average minimum distance between points is restricted by $N^{-1/d}$. However, since our purpose is to detect the existence of chaos, it already gives us sufficient evidence to determine the nature of the time series. There is no essential difference when applying this method to higher-dimensional cases. As examples, we use the time series generated by the Hénon map [10]

$$\begin{aligned} x_{n+1} &= 1 + y_n - ax_n^2 \\ y_{n+1} &= by_n \end{aligned} \tag{16}$$

where $a = 1.4$, and $b = 0.3$ (the $\{x_n\}$ series is used to reconstruct the state vector (x_n, x_{n+1})), and the more complicated Ikeda map [11, 12]:

$$\begin{aligned} x_{n+1} &= 1 + 0.9(x_n \cos\omega_n - y_n \sin\omega_n) \\ y_{n+1} &= 0.9(x_n \sin\omega_n + y_n \cos\omega_n) \\ \omega_n &= 0.4 - \frac{6}{1 + x_n^2 + y_n^2}. \end{aligned} \tag{17}$$

Figures 2(*b*) and (*c*) show that the results are similar to the one-dimensional logistic map case. (Note that the choice of data sets of 1000, 2000 and 3000 points for, respectively, the logistic map, the Hénon map and the Ikeda map is arbitrary but, in general, more data are needed for high-dimensional systems.) The slight variation of the slope for small ϵ values in figures 2(*b*) and (*c*) is probably caused by fluctuations due to insufficient data. This certainly appears to be the case when we use a larger data set (e.g. $N = 10\,000$), as shown in figure 2(*b*). For high-dimensional cases, the phenomenon which is worth pointing out is that if the embedding dimension is not large enough to unfold the attractor completely, we will not get the correct result. For example, (see figure 3), for the Ikeda map, if we reconstructed the attractor as a two-dimensional system, we do not get the linear relationship between $\ln n_{\epsilon,p}$ and p. (The false nearest neighbours method [8, 17] indicates that although we can unfold the attractor completely using $d_E = 4$, the percentage of the additional false nearest neighbours is very near to zero when d_E change from 3 to 4, so $d_E = 3$ is a good enough estimate [13].) Therefore, this method also provides a rough method to indicate an appropriate embedding dimension for the given data which is known to be chaotic. In figure 2(*d*), the identically independent distributed (i.i.d.) noise (uniformly distributed in $[0, 1]$) is used to test the method. As expected, $n_{p,\epsilon}$ drops with p dramatically. In this case, we can get some more detailed results. Since the noise signal $\{x_n\}$, $n = 1, 2, \ldots, N$ is distributed in $[0, 1]$, then for any arbitrarily picked reference point x_r, the probability for any x_n $(n \neq r)$ to drop into the ϵ-neighbourhood of x_r is ϵ. Thus the number of ϵ^0-neighbour points of x_r is $n(x_{r+0}, \epsilon) \sim \epsilon N$, and $n(x_{r+p}, \epsilon) \sim \epsilon^{p+1} N \sim n(x_{r+0})\epsilon^p$ for the ϵ^p-neighbour points. Therefore, the average number of ϵ^p-neighbours $\ln n_{\epsilon,p} \sim -p \ln \frac{1}{\epsilon}$. (For example, in figure 2(*d*), the expected slopes are -2.99 and -3.9 respectively for $\epsilon = 0.05$ and $\epsilon = 0.02$; the actual results from numerical simulations give -2.35 and -3.54.) This points to the conclusion that with typical small values of ϵ, the slope in the linear relationship for noise is very steep, and more importantly, the slope will change dramatically when we vary the value of ϵ, in contrast to the case of chaotic signals. We have also applied this method to the set of surrogate data from the logistic map. Since some correlations of the chaotic signal are preserved when we generate surrogate data, we are dealing with *coloured noise*. We use this term here to refer to noise which is not of the i.i.d. type (or white noise), and are not referring specifically to noise with power-law spectra (or so-called $1/f^\alpha$ noise [20–22]). The standard procedure [16] is followed: we first take the Fourier transform of the original data as used in figure 2(*a*), the phase is then randomized, and then the inverse Fourier transform is

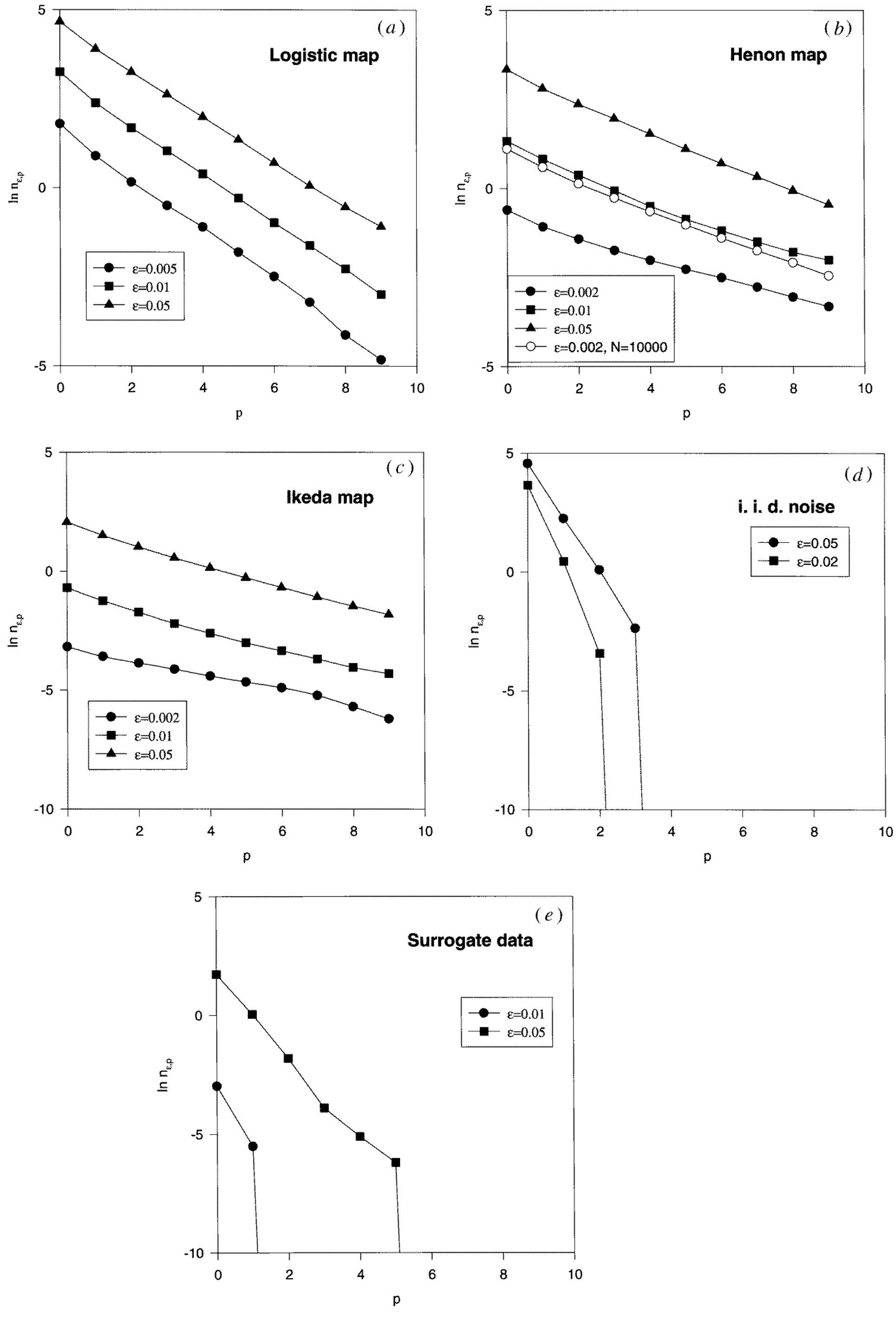

Figure 2. The relationship between $\ln n_{\epsilon,p}$ and p for various time series data. (*a*) Time series generated by the logistic map, equation (15). ($N = 1000$ points are used.) (*b*) Time series generated by the Hénon map, equation (16). Only $\{x_n, n = 0, 1, \ldots, N, N = 2000\}$ are used. The phase state is reconstructed by $\{x_n, x_{n+1}\}$. (*c*) Time series generated by the Ikeda map, equation (17). Only $\{x_n, n = 0, 1, \ldots, N, N = 3000\}$ are used and the phase state is reconstructed by $\{x_n, x_{n+1}, x_{n+2}\}$. (*d*) i.i.d. random series. ($N = 1000$ points are used.) (*e*) The surrogate data from the time series used in figure 1(*a*).

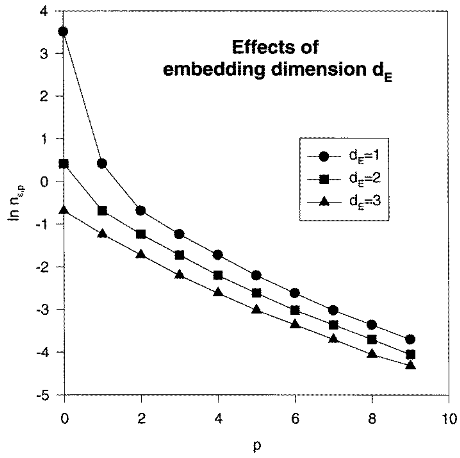

Figure 3. The influence of the embedding dimension for the Ikeda map (as in figure 1(*c*)). $\{x_n, n = 0, 1, \ldots, N\}$, $N = 3000$ are used.

taken to obtain the surrogate data. The resultant data are random but retains the variance and autocorrelation of the original time series. Figure 2(*e*) displays the typical result. It reflects clearly the characteristic of noise except for some minor differences; specifically, the linear scaling range (in which $\ln n_{\epsilon,p}$ and p have a linear relationship) is larger than that for the white noise case, but the slope still varies with different ϵ values (see figures 2(*d*) and (*e*), the case of $\epsilon = 0.05$). We believe that this is caused by the remaining autocorrelation in the surrogate data. We also found that in the case of $1/f^{\alpha}$ noise, the characteristic metioned above can still be observed. Further study is necessary to determine how the present method can discriminate between these different kinds of randomness. Most of the existing methods of analysis will encounter some unexpected difficulty when applied to the real experimental data. To examine the effectiveness of this method for real-world data, we have also applied it to some experimental data [23], which are determined to be chaotic time series based on other methods. 10 000 data points in the time series are used in our analysis. A segment of the time series is displayed in figure 4(*a*). We reconstructed the phase space with the parameters $d = 3$ and $\tau = 22$, which give the first minimum in the mutual information. The results are shown in figure 4(*b*), which indicates the chaotic nature of the time series. For comparison, we also displayed the result for the periodic time series which is generated in the same experiment with different parameters. We expect that the results will be slightly different when different embedding parameters (e.g. τ and d) are used. Because this is a continuous time series, general methods are still lacking for the best choice of time delay τ and embedding dimension d, and we cannot tell which is the best result. In figure 4(*b*) we just present one of the typical results. However, the qualitative characteristics are the same for different parameters: for periodic data, $\ln n_{\epsilon,p}$ is essentially flat as p is varied, whereas a positive KS entropy is strongly suggested in the chaotic case. We would like to point out that another advantage of this method is its behaviour with respect to the noise. As is well known, noise could lead to erroneous estimation of the fractal dimension and Lyponov exponents of the system, and could present various difficulties in the attempt to distinguish chaotic systems from stochastic ones. In our method, however, this problem can be partly solved, at least for i.i.d. noise. If an ϵ value is suitably chosen, the method will eliminate the effects of the noise. More specifically, suppose the variance of the noise is σ. If $\epsilon > \sigma$, the existence of noise will not influence the result (i.e. the slope of $\ln n_{p,\epsilon}$ versus p); and if $\epsilon < \sigma$, the effects of the noise will tend to increase the steepness in the slope. This can be understood as follows. Consider any point x_n in phase space. If the variance of the background noise is smaller then ϵ, it cannot kick out all of

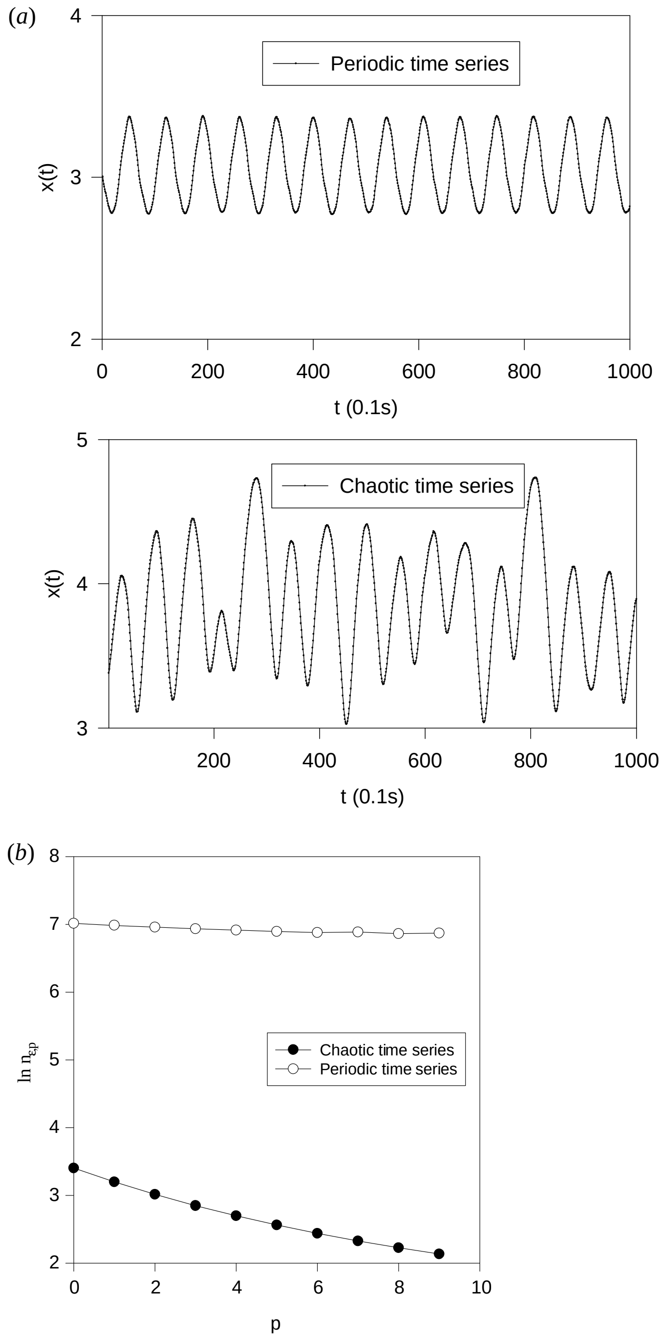

Figure 4. The results for the experimental data from [23]. $N = 10\,000$ data points are used and the phase space is reconstructed from recorded data with $d = 3$ and $\tau = 22$. (*a*) The original time series. (*b*) The relationship between $\ln n_{\epsilon,p}$.

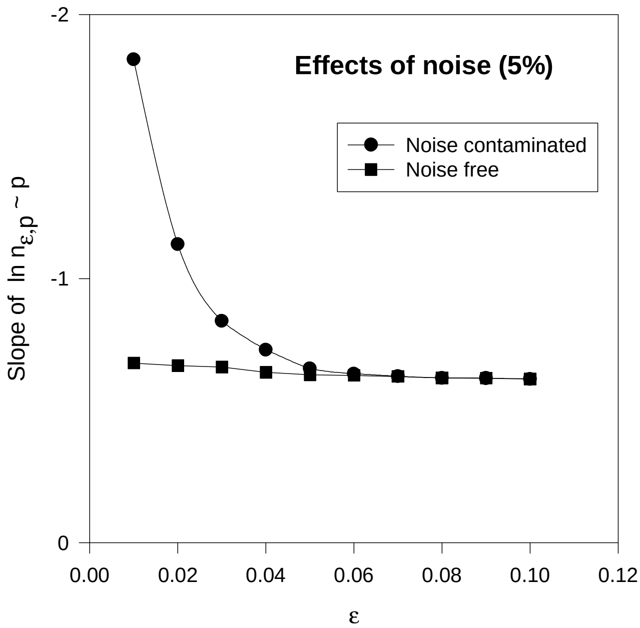

Figure 5. The effects of noise. The noisy data are obtained by adding 5% noise (evenly distributed within $[0, 0.05]$) to the time series used in figure 1(*a*). The slope is calculated at $\epsilon = 0.01, 0.02, \ldots, 0.1$. The results for clean data are also displayed for comparison.

ϵ^p-neighbour points of x_n, and a certain fraction of the true ϵ^p-neighbour points remain. Since the effect of the noise is identical everywhere, and the false neighbour points kicked in by noise in fact have no contribution when we go to larger values of p, the relationship between $\ln n_{\epsilon,p}$ and p will remain. However, for higher noise levels, since the true neighbour points will now be totally submerged under the noise, the result obtained will show the typical characteristics of noise. We show the behaviour of a chaotic system contaminated by noise in figure 5. In this case, the time series (generated by the logistic map) used in figure 1(*a*) is mixed with 5% noise. We computed the slopes of $\ln n_{\epsilon,p} \sim p$ at some different values of ϵ for both the noise-contaminated and noise-free cases. We can see clearly from the figure that the slope changes radically when ϵ is near the variance of the noise. The method proposed here thus provides a rough tool for estimating the signal-to-noise ratio through variation of the slope of $\ln n_{\epsilon,p} \sim p$ when varying ϵ, which may be useful for the detection of chaotic dynamics in a noisy environment. In conclusion, we propose here a new practical technique to characterize the chaotic dynamics using the local properties of chaotic systems. We have introduced the concept of the p-steps ϵ-neighbour points, and demonstrated that the average number of the ϵ^p-neighbours can be related to the KS entropy of the system. By investigating the evolution of the average number of the ϵ^p-neighbour points with the time step p, we can identify and differentiate low-dimensional chaotic systems. It is entirely based on the observed time series and can be easily generalized to higher dimensions. The computation is relatively simple. Furthermore, the method is robust to low-level noise, and may even be used to estimate the signal-to-noise ratio in the data.

Acknowledgment

GX acknowledges the research scholarship support at the National University of Singapore.

References

[1] Schuster H G 1988 *Deterministic Chaos: an Introduction* 2nd edn (Weinheim: Physik-Verlag)
[2] Grebogi C and Yorke J A 1997 *The Impact of Chaos on Science and Society* (New York: United Nation University Press)
[3] Abarbanel H D I, Brown R, Sidorowich J J and Lev Sh Tsimring 1993 *Rev. Mod. Phys.* **65** 1331
[4] Tanake T, Aihara K and Taki M 1996 *Phys. Rev.* E **54** 2122
[5] Packard N H, Crutchfield J P, Farmer J D and Shaw K S 1980 *Phys. Rev. Lett.* **45** 721
[6] Takens F 1981 Detecting strange attractors in Turbulence *Dynamical System and Turbulence (Lecture Notes in Mathematics vol 898)* (Berlin: Springer)
[7] Sauer T, Yorke J A and Casdagli M 1991 *J. Stat. Phys.* **65** 579
[8] Abarbanel H D I 1996 *Analysis of Observed Chaotic Data* (New York: Springer)
[9] Grassberger P and Procaccia I 1983 *Phys. Rev. Lett.* **50** 346
[10] Hénon N 1976 *Commun. Math. Phys.* **50** 69
[11] Ikeda K 1979 *Opt. Commun.* **30** 257
[12] Hammel S M, Jones C K R T and Moloney J V 1985 *J. Opt. Soc. Am.* B **4** 552
[13] Schroer C G, Sauer T, Ott E and Yorke J A 1998 *Phys. Rev. Lett.* **80** 1410
[14] Ott E, Sauer T and Yorke J A 1994 *Coping with Chaos* (New York: Wiley)
[15] Kantz H and Schreiber T 1997 *Nonlinear Time Series Analysis* (Cambridge: Cambridge University Press)
[16] Theiler J, Eubank S, Longtin A, Galdrikian B and Farmer J D 1992 *Physica* D **58** 77
[17] Kennel M B, Brown R and Abarbanel H D I 1992 *Phys. Rev.* A **45** 3403
[18] Crutchfield J P and Packard N H 1983 *Physica* D **7** 201
[19] Ledrappier F and Young L S 1985 *Ann. Math.* **122** 509
[20] Brin M and Katok A 1983 On the local entropy *Geometric Dynamics (Springer Lecture Notes vol 1007)* (Berlin: Springer)
[21] Osborne A R and Provenzale A 1989 *Physica* D **35** 357
[22] Theiler J 1991 *Phys. Lett.* A **155** 480
[23] Eric Weeks http://glinda.lrsm.upenn.edu/˜weeks/research/tseries1.html